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Moving Multi-Channel Systems in a Finite Volume

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The spectrum of a system with multiple channels composed of two hadrons with nonzero total momentum is determined in a finite volume with periodic boundary conditions using effective field theory methods. The results presented are accurate up to exponentially suppressed corrections in the volume. The formalism allows one to determine the phase shifts and mixing parameters of $\pi\pi - KK$ isosinglet coupled channels directly from Lattice Quantum Chromodynamics. We show that the extension to more than two channels is straightforward and present the result for the three channels. From the energy quantization condition, the volume dependence of electroweak matrix elements of two-hadron processes is extracted, for both relativistic and non-relativistic systems. In the NR case, we pay close attention to processes that mix the ${}^1S_0 - {}^3S_1$ two-nucleon states, e.g. proton-proton fusion ($pp \to d + e^+ + \nu_e$), and show how to determine the transition amplitude of such processes directly from lattice QCD.

Scattering processes in hadronic physics provide useful information about the properties of particles and their interactions. Some of these processes are well investigated in experiments with reliable precision. However, there are interesting two-body hadronic processes whose experimental determinations continue to pose challenges. They mainly include two-body hadronic scatterings near or above the kinematic threshold with the possibility of the occurrence of resonances. Here we discuss two pertinent cases in Quantum Chromodynamics (QCD), the first of which is the scalar sector, whose nature is still puzzling (See for example [1] and references therein). While some phenomenological models suggest the scalar resonances to be tetraquark states (as first proposed by Jaffe [2]), others propose these to be weakly bound mesonic molecular states. The most famous of which are the flavorless $a_0(980)$ and $f_0(980)$, which are considered to be candidates for a $K\bar{K}$ molecular states [3], [4], [5]. In order to shine a light on the nature of these isosinglet resonances, it would be necessary to perform model-independent multichannel calculations including the $\{\pi\pi, \pi\pi\pi\pi, K\bar{K}, \eta\eta\}$ scattering states directly from the underlying theory of QCD. In the baryonic sector, observations of the strong attractive nature of the isosinglet $\bar{K}N$ scattering channel led to the postulation of kaon condensation in dense nuclear matter [6]. However, extracting $\bar{K}N$ scattering parameters is a rather challenging task, due to the presence of the $\Sigma\pi$ scattering channel and the $\Lambda(1405)$ resonance bellow the $\bar{K}N$ threshold, see for example Ref. [7]. Previous chiral perturbation theory (χPT) calculations have found inconsistency between experimental determination of the $\bar{K}N$ scattering length from scattering data and kaonic hydrogen level shifts [8], [9], [10], [11], [12], but as with any low energy effective field theory (EFT) calculation, there are unaccountable systematic errors associated with the large number of unknown low energy coefficients (LECs) needed to perform accurate calculations of multichannel processes.

In addition to these strongly coupled scattering processes, there are weak processes involving multi-hadron states that require further investigation. For instance, Lattice QCD (LQCD) calculations have recently shown further evidence of $\Xi^-\Xi^-$ and $\Lambda\Lambda$ (H-dibaryon) shallow bound states [13], [14], [15]. This will certainly reignite experimental searches for evidences of these states. Among the possible weak decays of the H-dibaryon include $H \to (n\Lambda, n\Sigma^0, p\Sigma^-, nn)$ [16]. In hyper-nuclear physics, there has been much interest in a definitive determination of the contribution of non-mesonic weak decays ($\Lambda N \to NN$, $\Lambda NN \to NNN$) to the overall decay of hyper-nuclei. In particular, as discussed in [17] (and references therein), there has been a long standing puzzle regarding the theoretical underestimation of the ratio of the decay widths $\Gamma(\Lambda n \to nn)/\Gamma(\Lambda p \to np)$ as compared to the experimental value. Certainly a great deal of progress has been made by meson exchange models in order to close this gap, but a model-independent calculation directly form QCD would give further insight into the mechanism of these decays. These two cases illustrate processes where it is necessary to evaluate weak matrix elements between multi-hadronic states.

Currently, LQCD provides the most reliable option for performing calculations of low energy QCD observables. LQCD calculations are necessarily performed in a Eucledian and finite spacetime volume. Although the former

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forbids one to calculate the physical scattering amplitudes from their Euclidean counterparts away from the kinematic threshold due to the Maiani-Testa theorem [18], the latter is proven to be a useful tool in extracting the physical scattering quantities from lattice calculations. In his prominent work, Lüscher showed how one can obtain the infinite volume scattering phase shifts by calculating energy levels of two-body scattering states in the finite volume [19], [20]. The Lüscher method which was later generalized to the moving frames in Ref. [21–23], is only applicable to scattering processes below the inelastic threshold. Therefore it cannot be used near the inelastic threshold where new channels open up, and a generalized formalism has to be developed to deal with the coupled(multi)-channel processes. A direct calculation of the near threshold scattering quantities using LQCD can lead to the identification of resonances in QCD such as those discussed above, and provide reliable predictions for their masses and their decay width. One such generalization was developed by Liu *et al.* in the context of quantum mechanical two-body scattering [24], [25]. There, the authors have been able to deduce the relation between the infinite volume coupled channel S-matrix elements and the energy shifts of the scattering particles in the finite volume by solving the coupled Schrodinger equation both in infinite volume and on a torus. The idea is that as long as the exponential volume corrections are sufficiently small, the polarization effects, as well as other field theory effects, are negligible. Therefore after replacing the non-relativistic dispersion relations with their relativistic counterparts, the quantum mechanical result of Liu *et al.* [24], [25] is speculated to be applicable to the massive field theory. In another approach, Lage *et al.* considered a two-channel Lippman-Schwinger equation in a non-relativistic effective field theory (NR EFT). They presented the mechanism for obtaining the $\bar{K}N$ scattering length, and studying the nature of the $\Lambda(1405)$ resonance from LQCD [26]. Later on, Bernard *et al.* generalized this method to the relativistic EFT which would be applicable for coupled $KK-\pi\pi$ channels [1]. Unitarized chiral perturbation theory (UCHPT) provides another tool to study a variety of resonances in the coupled channel scatterings. This method uses the Bethe-Salpeter equation for a coupled-channel system to dynamically generate the resonances in both light meson sector and meson-baryon sector in the infinite volume, see for example [27], [28], [29], [30]. When applied in the finite volume, the volume-dependent discrete energy spectrum can be produced, and by fitting the parameters of the chiral potential to the measured energy spectrum on the lattice, the resonances can be located by solving the scattering equations in the infinite volume. This method has been recently used to study the resonances $f_0(600)$, $f_0(980)$ and $a_0(980)$ in [31], [32], $\Lambda(1405)$ in [33], $a_1(1260)$ in [34], $\Lambda_c(2595)$ in [35], and $D_{s^*0}(2317)$ in [36] in the finite volume. One should note that in contrast to the single channel scattering in the finite volume, the coupled-channel scattering requires determination of three independent scattering parameters which would require in the very least three measurements of the energy levels in the finite volume. As proposed in [1], [31], one can impose twisted boundary condition in the lattice calculation to be able to increase the number of measurements by varying the twist angle and further constrain the scattering parameters. Another tool to circumvent this problem is the use of asymmetric lattices as is investigated in [1], [31], [33]. Alternatively, one can perform calculations with different boost momenta [33], [34].

The goal of this paper is two-fold. First, we present a model-independent fully relativistic framework for determining the finite volume (FV) coupled-channel spectrum in a moving frame. Secondly, we show how to extract current-operator matrix elements in the two hadron sector directly from LQCD, for both relativistic and non-relativistic processes. These two are small stepping-stones towards one of the overarching goals of hadronic physics, which is to determine properties of multi-hadron system directly from the underlying fundamental theory of QCD.

This paper is structured as follows. In the first section we present the result from a scalar field theory model, which illustrates all the features of the problem at hand and allows us to derive the quantization condition (QC) using a diagrammatic expansion. Although, in deriving the QC for the toy model we make a series of approximations, it is shown that this result is in perfect agreement with the exact QC which is obtained from the generalization of the work by Kim *et al.* [22], [37]. In section I B, we derive the general form of the finite volume quantization condition for N arbitrarily strongly coupled two-body states. This result has been independently derived and confirmed in a parallel work by Hansen and Sharpe [38]. In this paper, most of the emphasis will be placed on the N=2 case, but the result for the N=3 case will be also explicitly shown.

After developing the finite volume coupled-channel formalism, we extend our work to be able to determine electroweak matrix elements in the two hadron sector. The formalism for extracting the physical transition amplitude for $K\to\pi\pi$ from the finite volume matrix elements of the weak Hamiltonian in the finite volume, has been developed by Lellouch and Lüscher [39]. This former quantity has been shown to be proportional to the latter at the leading order in the weak coupling, and the proportionality factor, the so-called LL factor, is shown to be related to the derivative of the $\pi\pi$ scattering phase shift. The generalization of the LL factor to the moving frame is given in [22], [37]. Ultimately, it would be desirable to calculate electroweak matrix elements of states containing an arbitrary number of hadrons, but for the time being we restrict ourselves to the two body sector. Nevertheless, the two body sector encompasses a large number of interesting processes including those discussed above. Additionally, as is discussed in [40], the weak disintegration of deuteron in processes such as $\nu d\to\nu pn$ and $\nu_e d\to nne^+$ is of great importance in

neutrino experiments. These processes entail an electroweak mixing between ${}^1S_0 -{}^3S_1$ two-nucleon channels, and for energies bellow the pion-production threshold the dynamics of this system can be described using an EFT without pions ($EFT\,(\not{\pi})$) [41], [42], [43]. One hopes to get the physical weak transition amplitude in these processes from a LQCD calculation. By applying two alternative methods, degenerate perturbation theory and the density of states method, we give a generalization of the LL factor for $2 \to 2$ weak processes in a moving frame. This includes the derivation of the relativistic LL factor in the mesonic sector as well as its non-relativistic extension for the baryonic sector. Finally the effect of one-body weak interactions on the finite volume quantization condition and on the LL factor is discussed.

I. MESON-MESON COUPLED CHANNELS

The goal of this section is to present the quantization condition for N coupled-channel system in a moving frame. We present two independent ways to obtain the quantization condition desired. In section I A, we present a toy model that illustrates the features of the problem at hand and allows us to determine the scattering matrix using a diagrammatic expansion. In order to derive the quantization using this approach, it is convenient to consider the case where the parameter responsible for mixing the two channels is small. However, as shown in section I B, the result presented is exact for an arbitrary mixing parameter.

In section I B, we show how to obtain the quantization condition for N arbitrarily strongly coupled two-body channels in a moving frame by solving the Bethe-Salpeter equation. This case is the most relevant if we want to consider systems coupled via QCD interactions, e.g. the $I = 0$ system $\pi\pi \to (K\bar{K}, \eta\eta) \to \pi\pi$ [1].

A. Toy problem: two weakly coupled two-meson channels in the moving frame

In this section, we consider a two-meson coupled system with J=0 angular momentum. The two channels will be labeled I and II. In general the four mesons can have different masses and quantum numbers, but for the time being we restrict ourselves to the case where the two mesons in each channel are identical. Using the "barred" parameterization [44], the S-matrix describing this system can be written as

$$S_2 = \begin{pmatrix} e^{i2\delta_I}\cos 2\bar{\epsilon} & ie^{i(\delta_I+\delta_{II})}\sin 2\bar{\epsilon} \\ ie^{i(\delta_I+\delta_{II})}\sin 2\bar{\epsilon} & e^{i2\delta_{II}}\cos 2\bar{\epsilon} \end{pmatrix}. \tag{1}$$

where δ_I and δ_{II} are the phase shifts corresponding to the scattering in channels I and II respectively, and $\bar{\epsilon}$ is a parameter which characterizes the mixing between the channels. The subscript 2 on S denotes the number of coupled channels.

At energies below the four-meson threshold, the dynamics of such system can be described by a simple scalar effective field theory (EFT)

$$\mathcal{L} = \sum_{i=I,II} \phi_i^\dagger(\partial^2 - m_i^2)\phi_i + \begin{pmatrix} \frac{\phi_I\phi_I}{2} & \frac{\phi_{II}\phi_{II}}{2} \end{pmatrix}^\dagger \begin{pmatrix} c_I & g/2 \\ g/2 & c_{II} \end{pmatrix} \begin{pmatrix} \phi_I\phi_I/2 \\ \phi_{II}\phi_{II}/2 \end{pmatrix} + \cdots \tag{2}$$

where ϕ_i is the meson annihilation operator for the i^{th} channel, and c_i and g are the LECs of the theory. Ellipsis denotes higher derivative four-meson terms which will be neglected in this section. Our goal is to determine the coupled-channel spectrum in a finite volume (FV) that falls within the p-regime of LQCD [45], [46], defined by $\frac{m_\pi L}{2\pi} \gg 1$ where m_π is the pion mass, and L is the spatial extent of the volume. In this regime, finite volume corrections to the single particle dressed propagator are exponentially suppressed [19], as are contributions from t,u-channel scattering diagrams [19], [20, 39]. The leading order (LO) volume effects in the two-body sector arise from the presence of poles in the s-channel scattering diagrams. These lead to power-law volume corrections to the 2-particle spectrum [19], [20]. Therefore in the following discussion we will restrict ourselves to the contribution of such diagrams to the scattering matrix $\mathcal{M}$, whose $\mathcal{M}_{ij}$ matrix element corresponds to the scattering amplitude from the i^{th} channel to the j^{th} channel. Certainly, the t,u-channel diagrams contribute to the renormalization of the theory

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[1] It is important to mention that we are not making any claims about the relevance of the four-pion channel in the light-scalar sector of QCD. At this point, it is not evident how to incorporate such states into the calculation, therefore if one would choose to use the formalism presented here to study the light-scalar sector of QCD, there will be an overall systematic error that must be accounted for through other means.

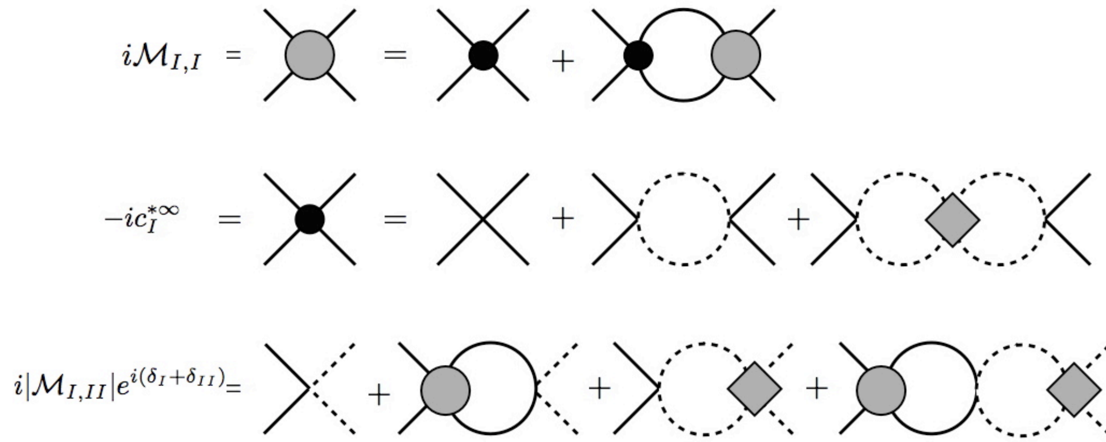

FIG. 1: Shown are the diagrammatic representations of the full scattering amplitudes. Solid (dashed) lines represent single particle propagators of the I (II) channel. The first two diagrams require the LECs to reproduce the scattering amplitude for the I channel. The grey circle(diamond) is the full scattering amplitude of the $I(II)$ channel. The black dot denotes the effective coupling for the I channel, which includes an infinite series of intermediate II bubble diagrams. There are two other diagrams for the II channel that require the LECs to recover $\mathcal{M}_{II,II}$ and have not shown. The third diagram ensures that the mixing term g reproduces the off-diagonal scattering matrix elements.

and therefore to the definition of the LECs. But for energies bellow the inelastic threshold, their finite volume effects are suppressed. In the non-relativistic (NR) limit, the t,u-channel diagrams are prohibited, so at low energies it is sensible to neglect their contributions. Alternatively, one can redefine the LECs to absorb the contributions of the t,u-channel diagrams.

When considering momentum-independent interactions, the infinite volume loops contributing to the s-channel diagrams are

$$G_i^{\infty} \equiv \frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{[(k-P)^2-m^2+i\epsilon]\,[k^2-m^2+i\epsilon]}, \tag{3}$$

where $P=(E,\mathbf{P})$ is the total four-momenta of the system. It is straightforward to write down the equations that $\{c_I, c_{II}, g\}$ must simultaneously satisfy in order to reproduce the $\mathcal{M}$-matrix elements of the theory, similar to the approach of [47] in examining the role of $\Delta\Delta$ intermediate states in the ${}^1S_0$ NN scattering. These equations, which are diagrammatically shown in Fig. (1), can be written as follows

$$c_i^{*\infty} = c_i + g^2\sum_{j\neq i}\frac{G_j^{\infty}}{1-c_j^* G_j^{\infty}} = \frac{-\mathcal{M}_{i,i}}{1-\mathcal{M}_{i,i}G_i^{\infty}}, \qquad g = |\mathcal{M}_{I,II}|e^{i(\delta_I+\delta_{II})}(1-c_I^* G_I^{\infty})(1-c_{II}^* G_{II}^{\infty}) \tag{4}$$

where $\mathcal{M}_{i,i}$ is the full relativistic S-wave scattering amplitude for the i^{th} channel, and $\mathcal{M}_{I,II}$ is the amplitude responsible for mixing the two channels. Solving these coupled equations leads to a renormalized theory.

Once this is done, one can study physics in a finite volume. In particular, we are interested in the energy-eigenvalues of the meson-meson system placed in a finite lattice with periodic boundary conditions. The spectrum can be determined by requiring the real part of the inverse of the finite volume scattering amplitude to vanish. In a periodic lattice, the integrals over the spatial momenta appearing in Feynman diagrams are replaced by a sum over discretized three-momenta. In finite volume the integral in Eq. (3) is replaced by

$$G_i^{V} \equiv \frac{i}{2L^3}\sum_{\mathbf{k}}\int \frac{dk^0}{2\pi}\frac{1}{[(k-P)^2-m^2+i\epsilon]\,[k^2-m^2+i\epsilon]}, \tag{5}$$

where the spatial momenta are quantized due to the periodic boundary conditions $\mathbf{k}=2\pi\mathbf{n}/L$ for $\mathbf{n}\in Z^3$, while the temporal extent of our Minkowski space remains infinite. This sum suffers of the same UV divergence of Eq. (3), and the difference of the two, $\delta G_i^V \equiv G_i^V - G_i^{\infty}$, is finite. It is simplest to consider the case where the mixing term, g is small, and keep our expressions to leading order in g. Using this and the definitions of the LECs in Eq. (4), the finite

volume scattering amplitude of the I channel can be written as

$$(\mathcal{M}_{I,I})_V \approx \frac{i\mathcal{M}_{I,I}}{1+\mathcal{M}_{I,I}\delta G_I^V + \delta G_{II}^V \frac{|\mathcal{M}_{I,II}^2|e^{i2(\delta_I+\delta_{II})}}{\mathcal{M}_{I,I}(1+\mathcal{M}_{II,II}\delta G_{II}^V)}}. \tag{6}$$

Finally, we obtain the quantization condition for the coupled channel problem at LO in the mixing parameter

$$\mathcal{R}e\left\{|\mathcal{M}_{I,II}|^2 e^{i2(\delta_I+\delta_{II})} - \left(\mathcal{M}_{I,I}+\frac{1}{\delta G_I^V}\right)\left(\mathcal{M}_{II,II}+\frac{1}{\delta G_{II}^V}\right)\right\} = 0. \tag{7}$$

At this point we have refrained from using explicit expressions from the FV integrals and scattering matrix elements, these details will be presented in the following sections. It is important to note that despite the simplicity of this toy model, it illustrates all the features of the problem at hand, and, as it will be shown in the next section, the result for the weakly coupled channels presented above is the same to the strongly coupled case when only the s-wave contribution to the two-body scattering in the cubic lattice is taken into account.

B. N arbitrarily strongly coupled two-body channels in a moving frame

In the previous section, the following assumptions have been made. First of all, the higher order four-meson terms in the derivative expansion of the effective Lagrangian have been neglected. Secondly, the scattering is restricted to two coupled channels composed of identical particles, and only the s-wave scattering is considered. Most importantly, the mixing term was assumed to be small in deriving the quantization condition. As mentioned earlier, the latter condition is the most relevant when considering systems coupled via QCD interactions. In this section we will simultaneously remove all of this assumptions.

Most of the details associated with generalizing to a moving frame have been developed by Kim *et al.* [22], [37], which we will briefly review for completeness. A system with total energy and momentum, E and P, in the laboratory frame has a CM energy $E^* = \sqrt{E^2 - P^2}$. For the i^{th} channel with two mesons each having masses $m_{i,1}$ and $m_{i,2}$, the CM relative momentum is

$$q_i^{*2} = \frac{1}{4}\left(E^{*2} - 2(m_{i,1}^2 + m_{i,2}^2) + \frac{(m_{i,1}^2 - m_{i,2}^2)^2}{E^{*2}}\right), \tag{8}$$

which simplifies to $\frac{E^{*2}}{4} - m_i^2$ when $m_{i,1} = m_{i,2} = m_i$. Including all possible momentum dependent operators in the EFT, the moving frame FV loop one needs to evaluate is [22], [37]

$$G_i^V \equiv \frac{in_i}{L^3}\sum_{\mathbf{k}}\int \frac{dk^0}{2\pi}\frac{f(\mathbf{k})}{[(k-P)^2 - m_{i,1}^2 + i\epsilon][k^2 - m_{i,2}^2 + i\epsilon]} \equiv G_i^\infty + \delta G_i^V \tag{9}$$

where n_i is 1/2 if the particles in the i^{th} loop are identical and 1 otherwise. The function $f(\mathbf{k})$ holds the momentum dependance arising from the infinite tower of operators inserted on either side of the loop, i.e. the Bethe-Salpeter kernel, $\mathcal{K}$. The FV part of the sum can be written as

$$\delta G_i^V = in_i\left(\frac{q^* f_{00}^*(q_i^*)}{8\pi E^*} - \frac{i}{2E^*}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{lm}^*(q_i^*)c_{lm}^P(q_i^{*2})\right), \tag{10}$$

where f_{lm} is defined via the spherical harmonic (Y_{lm}) decomposition of $f(\mathbf{k}^*)$,

$$f(\mathbf{k}^*) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{lm}(k^*)k^{*l}\sqrt{4\pi}Y_{lm}(\theta^*,\phi^*). \tag{11}$$

The function $c_{lm}^P(q_i^{*2})$ can be written in terms of the three-dimensional Zeta function, $\mathcal{Z}_{lm}^d$,

$$c_{lm}^P(q^{*2}) = -\frac{\sqrt{4\pi}}{\gamma L^3}\left(\frac{2\pi}{L}\right)^{l-2}\mathcal{Z}_{lm}^d[1;(q^*L/2\pi)^2], \qquad \mathcal{Z}_{lm}^d[s;x^2] = \sum_{\mathbf{r}\in P_d}\frac{Y_{l,m}(\mathbf{r})}{(r^2-x^2)^s} \tag{12}$$

where the sum is performed over $P_d = \left\{\mathbf{r}\in\mathbf{R}^3 \mid \mathbf{r} = \gamma^{-1}(\mathbf{m}_{||} - \alpha\mathbf{d}) - \mathbf{m}_\perp, m\in\mathbf{Z}\right\}$, $\mathbf{d}$ is the normalized boost vector $\mathbf{d} = \mathbf{P}L/2\pi$, the relativistic γ is defined by $\gamma = E/E^*$, and $\alpha = \frac{1}{2}\left(1 + \frac{m_1^2 - m_2^2}{E^{*2}}\right)$ with $E^* = \sqrt{q^* + m_1^2} + \sqrt{q^* + m_2^2}$ [48], [49], [50].

The generalization of the quantization conditions for N channels that are coupled via an arbitrarily strong interaction is straightforward by upgrading the Bethe-Salpeter kernel, $\mathcal{K}$, to not just be a matrix over angular momentum but also over N channels. Similarly, it is convenient to represent the N loops as a diagonal matrix $\mathcal{G} = diag(\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_N)$. The kernel is assured to reproduce the infinite volume scattering matrix ($\mathcal{M}$) by solving the following matrix equation

$$i\mathcal{M} = -i\mathcal{K} - i\mathcal{K}\mathcal{G}^{\infty}\mathcal{K} - i\mathcal{K}\mathcal{G}^{\infty}\mathcal{K}\mathcal{G}^{\infty}\mathcal{K} + \cdots = -i\mathcal{K}\frac{1}{1-\mathcal{G}^{\infty}\mathcal{K}} \Rightarrow \mathcal{K} = -\mathcal{M}\frac{1}{1-\mathcal{G}^{\infty}\mathcal{M}}. \tag{13}$$

With this definition of the kernel, one can proceed to evaluate poles of the N-channels FV scattering matrix by replacing the infinite volume loops $\mathcal{G}^{\infty}$ with their finite $\mathcal{G}^{V}$ counterparts,

$$-i(\mathcal{M})_V = -i\mathcal{K} - i\mathcal{K}\mathcal{G}^{V}\mathcal{K} - i\mathcal{G}^{V}\mathcal{K}\mathcal{G}^{V}\mathcal{K} + \cdots = -i\mathcal{K}\frac{1}{1-\mathcal{G}^{V}\mathcal{K}} = -i\frac{1}{1-\mathcal{M}\mathcal{G}^{\infty}}\mathcal{M}\frac{1}{1+\delta\mathcal{G}^{V}\mathcal{M}}(1-\mathcal{M}\mathcal{G}^{\infty}). \tag{14}$$

Finally arriving at the quantization condition

$$\mathcal{R}e\left\{\det(1+\delta\mathcal{G}^{V}\mathcal{M})\right\} = 0 \tag{15}$$

where $\mathcal{M}$ is evaluated on-shell. For N=1, one reproduces the result first obtained by Rummukainen and Gottlieb [21] and later confirmed by Kim *et al.* [22] as follows. First note that it is convenient to evaluate the determinant using the spherical harmonic basis of $\delta\mathcal{G}^{V}$ and the on-shell scattering amplitude $\mathcal{M}_i$ [22]

$$(\delta\mathcal{G}_i^{V})_{l_1,m_1;l_2,m_2} = i\frac{q_i^* n_i}{8\pi E^*}\left(\delta_{l_1,l_2}\delta_{m_1,m_2} - i\frac{4\pi}{q_i^*}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{\sqrt{4\pi}}{q_i^{*l}}c^{P}_{lm}(q_i^{*2})\int d\Omega^* Y^*_{l_1m_1}Y^*_{lm}Y_{l_2m_2}\right) \tag{16}$$

$$(\mathcal{M}_i)_{l_1,m_1;l_2,m_2} = \delta_{l1,l2}\delta_{m1,m2}\frac{8\pi E^*}{n_i q_i^*}\frac{e^{2i\delta_i^{(l)}(q_i^*)}-1}{2i}. \tag{17}$$

If the two meson interpolating operator is in the A_1^+ irreducible representation of the cubic group, the energy eigenstates of the system have overlap with the $l = 0, 4, 6, \ldots$ angular states at zero total momentum, making the truncation at $l_{max} = 0$ a rather reasonable approximation. When $\mathbf{P} \neq 0$, the symmetry group is reduced, and at low energies the $l = 0$ and $l = 2$ partial wave do mix [21]. Nevertheless, if one truncates the determinant over the angular momentum at $l_{max} = 0$, the familiar quantization condition

$$q_i^* \cot(\delta_i^0) = \frac{2}{\gamma L\sqrt{\pi}}\mathcal{Z}^{d}_{00}[1; (q_i^* L/2\pi)^2]. \tag{18}$$

is recovered. It is convenient to introduce a pseudo-phase defined by

$$q_i^* \cot(\phi_i) \equiv 4\pi c^{P}(q_i^*). \tag{19}$$

to rewrite the quantization condition as:

$$\cot(\phi_i) = -\cot(\delta_i) \Rightarrow \delta_i + \phi_i = m\pi, \tag{20}$$

where m is an integer.

For the N=2, the expression for the scattering amplitude in Eq. (17) is modified, as it now depends on the mixing angle $\bar{\epsilon}$, and the scattering matrix is no longer diagonal, while still symmetric. The off-diagonal terms are labeled $\mathcal{M}_{I,II}$. Using the definition of the S-matrix for the coupled-channel system, Eq. (1), the scattering matrix elements can be written as

$$(\mathcal{M}_{i,i})_{l_1,m_1;l_2,m_2} = \delta_{l_1,l_2}\delta_{m_1,m_2}\frac{8\pi E^*}{n_i q_i^*}\frac{\cos(2\bar{\epsilon})e^{2i\delta_i^{(l)}(q_i^*)}-1}{2i} \tag{21}$$

$$(\mathcal{M}_{I,II})_{l_1,m_1;l_2,m_2} = \delta_{l_1,l_2}\delta_{m_1,m_2}\frac{8\pi E^*}{\sqrt{n_I n_{II} q_I^* q_{II}^*}}\sin(2\bar{\epsilon})\frac{e^{i(\delta_I^{(l)}(q_I^*)+\delta_{II}^{(l)}(q_{II}^*))}}{2}. \tag{22}$$

Where the usual relativistic normalization of the states is used in evaluating the S-matrix element. From Eq. (15) one obtains

$$\mathcal{R}e\left\{\det\begin{pmatrix}1+\delta\mathcal{G}_I^{V}\mathcal{M}_{I,I} & \delta\mathcal{G}_I^{V}\mathcal{M}_{I,II}\\ \delta\mathcal{G}_{II}^{V}\mathcal{M}_{I,II} & 1+\delta\mathcal{G}_{II}^{V}\mathcal{M}_{II,II}\end{pmatrix}\right\} = 0, \tag{23}$$

where the determinant is not only over the number of channels but also over angular momentum which is left implicit. In deriving this result we have made no assumption about the relative size between the scattering matrix elements, but when $l_{max}=0$ we recover the LO result in Eq. (7). For $l_{max}=0$ one can show that the determinant has in fact no imaginary part, and using the pseudo-phase definition in Eq. (19), the quantization condition can be rewritten in a manifestly real form:

$$\cos 2\bar{\epsilon}\cos\left(\phi_1+\delta_1-\phi_2-\delta_2\right)=\cos\left(\phi_1+\delta_1+\phi_2+\delta_2\right). \tag{24}$$

One can see that in the $\epsilon\to 0$ the decoupled quantization conditions for channels I and II , Eq. (20) are recovered. The extension to a larger number of coupled-channels is straightforward. As an example, we consider the N=3. Unitarity allows us to parametrize the S-matrix using three phases shifts $\{\delta_I,\delta_{II},\delta_{II}\}$ and three mixing angles $\{\bar{\epsilon}_1,\bar{\epsilon}_2,\bar{\epsilon}_3\}$

$$S_3=\begin{pmatrix} e^{i2\delta_I}c_1 & ie^{i(\delta_I+\delta_{II})}s_1c_3 & ie^{i(\delta_I+\delta_{III})}s_1s_3\\ ie^{i(\delta_I+\delta_{II})}s_1c_2 & e^{i2\delta_{II}}\left(c_1c_2c_3-s_2s_3\right) & ie^{i(\delta_I+\delta_{III})}\left(c_1c_2s_3+s_2c_3\right)\\ ie^{i(\delta_I+\delta_{III})}s_1s_2 & ie^{i(\delta_{II}+\delta_{III})}\left(c_1s_2c_3+c_2s_3\right) & ie^{i2\delta_{III}}\left(c_1s_2s_3-c_2c_3\right)\end{pmatrix}. \tag{25}$$

where $c_i=\cos(2\bar{\epsilon}_i)$, $s_i=\sin(2\bar{\epsilon}_i)$. Note that in the limit $\epsilon_2=\epsilon_3=0$ the third channel decouples, and one obtains a block diagonal matrix composed of S_2 corresponding to the $I-II$ coupled channel, as well as a single element corresponding to the scattering in the uncoupled channel III. The spectrum of three-coupled channel is defined by

$$\mathcal{R}e\left\{\det\begin{pmatrix}1+\delta\mathcal{G}_I^V\mathcal{M}_{I,I} & \delta\mathcal{G}_I^V\mathcal{M}_{I,II} & \delta\mathcal{G}_I^V\mathcal{M}_{I,III}\\ \delta\mathcal{G}_{II}^V\mathcal{M}_{II,I} & 1+\delta\mathcal{G}_{II}^V\mathcal{M}_{II,II} & \delta\mathcal{G}_{II}^V\mathcal{M}_{II,III}\\ \delta\mathcal{G}_{III}^V\mathcal{M}_{III,I} & \delta\mathcal{G}_{III}^V\mathcal{M}_{III,II} & 1+\delta\mathcal{G}_{III}^V\mathcal{M}_{III,III}\end{pmatrix}\right\}=0, \tag{26}$$

where the scattering matrix elements can be determined from Eq. (25) using the relationship between the scattering amplitudes and the S-matrix elements $(\mathcal{M}_{i,j})_{l_1,m_1;l_2,m_2}=\delta_{l_1,l_2}\delta_{m_1,m_2}8\pi E^*((S_3^{(l_1)})_{i,j}-\delta_{i,j})/(2i\sqrt{n_in_jq_i^*q_j^*})$.

II. TWO-BODY ELECTROWEAK MATRIX ELEMENTS IN FV

As discussed in the introduction, electroweak processes in the two-hadron sector of QCD encompass a variety of interesting physics, so it would be desirable to calculate the electroweak matrix elements directly from lattice QCD. One of the very first attempts to develop a formalism for such processes from a finite volume Euclidean calculation is due to Lellouch and Lüscher. In their seminal work [39], they restricted themselves to $k\to\pi\pi$ decay in the kaon's rest frame, and showed that the absolute value of the transition matrix element in an Euclidian FV is proportional to the physical transition matrix element. This proportionality factor is known as the LL-factor. This formalism was then generalized to moving frames by Kim *et al*. [22]. Here we present the generalization of Lellouch and Lüscher formalism to processes where the initial and final states are composed of two-hadrons S-wave states. In the relativistic case, the coupled channels result Eq. (7) is used to derived the $2\to 2$ LL-factor for boosted systems. For the NR case, we use low energy EFT($\not\pi$) in two cases. The first case that is most pertinent for nuclear physics, regards the ${}^1S_0-{}^3S_1$ mixing which requires including one-body contributions to the weak matrix. The relationship between the FV matrix element and the physical transition amplitude is then found. It has been shown that the absolute value of these two are no longer proportional to each other as a result of one-body weak contributions. We then also present the result for weak processes which do not involve any one-body contributions.

A. Relativistic 2-Body LL-Factor

1. *Degenerate perturbation theory derivation*

In order to derive the relativistic two-body LL-factor, one first notes that in the absence of the weak interaction, the two states decouple, as a result the S-matrix becomes diagonal. As is pointed out by Lellouch and Lüscher, there is a simple trick to obtain the desired relation by assuming the initial and final states to be nearly degenerate with energy E_0^* (each satisfying Eq. (20)) when there is no weak interactions. Once the perturbative weak interaction is turned on, the degeneracy is lifted, and the energy eigenvalues are

$$E_\pm^*=E_0^*\pm V|\mathcal{M}_{I,II}^V|\equiv E_0^*\pm\Delta E, \tag{27}$$

where $\mathcal{M}^V_{I,II}$ is the FV matrix element of the weak Hamiltonian density. Consequently, the CM momenta and the scattering phase shifts acquire perturbative corrections of the form

$$\Delta q_i^* = \frac{1}{4q_i^*}\left(E_0^* - \frac{(m_{i,1}^2 - m_{i,2}^2)^2}{E_0^{*3}}\right) V|\mathcal{M}^V_{I,II}| \equiv \Delta\tilde{q}_i^*\ V|\mathcal{M}^V_{I,II}| \tag{28}$$

and

$$\Delta\delta_i(q_i^*) = \delta_i'(q_i^*)\Delta\tilde{q}_i^*\ V|\mathcal{M}^V_{I,II}|. \tag{29}$$

where $\delta_i'(q_i^*)$ denotes the derivative of the phase shift with respect to the momentum evaluated at the free CM momentum. The perturbed energy necessarily satisfies the quantization condition Eq. (23). The generalized LL-factor for $2 \to 2$ scattering is then obtained by Taylor expanding Eq. (23) to leading order in the weak matrix element about the free energy solution,

$$|\mathcal{M}^\infty_{I,II}|^2 = V^2\left\{\Delta\tilde{q}_I^*\Delta\tilde{q}_{II}^*\left(\frac{8\pi E_0^*}{n_I q_I^*}\right)\left(\frac{8\pi E_0^*}{n_{II} q_{II}^*}\right)(\phi_I'(q_I^*) + \delta_I'(q_I^*))\,(\phi_{II}'(q_{II}^*) + \delta_{II}'(q_{II}^*))\right\}|\mathcal{M}^V_{I,II}|^2. \tag{30}$$

where $\phi_i'(q_i^*)$ denotes the derivative of the pseudo-phase with respect to the momentum evaluated at the free CM momentum.

2. *Density of states derivation*

In the previous section, we arrived at the generalization of the LL factor for two-body matrix elements using the degeneracy of states argument. Lin *et al.* [51] showed that the the LL-factor for $K \to \pi\pi$ can also be derived using the density of states in the large volume limit, and this argument was then generalized by Kim *et al.* [22] to boosted systems. Here it will be shown that the results in Eq. (30) is also consistent with the work by Kim *et al.* Let $\sigma_i(\mathbf{x},t)$ be the two-particle annihilation operator for the i^{th} channel. Then the two particle correlation function in FV can be written as

$$\begin{aligned} C^V_{\sigma_i}(t) \ &\equiv\ \int d^3x\, e^{i\mathbf{P}\cdot\mathbf{x}}\langle 0|\sigma_i(\mathbf{x},t)\,\sigma_i^\dagger(\mathbf{0},0)|0\rangle_V = V\sum_m e^{-E_m t}\,|\langle 0|\sigma(\mathbf{0},0)|i;\mathbf{P},m\rangle_V|^2 \\ &\overset{L\to\infty}{\longrightarrow} V\int dE\rho_{V,i}(E)e^{-Et}\,|\langle 0|\sigma(\mathbf{0},0)|i;\mathbf{P},E\rangle_V|^2 . \end{aligned} \tag{31}$$

In the first equality we have inserted a complete set of states. In the second equality, we have introduced the density of states for the i^{th} channel, $\rho_{V,i}(E)$, which is defined as $\rho_{V,i}(E) = dm_i/dE$. Using Eqs. (20), (28) the density of states can be written as $\rho_{V,i}(E^*) = (\phi_i'(q_i^*) + \delta_i'(q_i^*))\,\Delta\tilde{q}_i^*/\pi$. In infinite volume the two-particle correlation function is [51]

$$C^\infty_{\sigma_i}(t) = \frac{n_i}{8\pi^2}\int dE\frac{q_i^*}{E^*}e^{-Et}\,|\langle 0|\sigma(\mathbf{0},0)|i;\mathbf{P},E\rangle_\infty|^2 , \tag{32}$$

where the factor of n_i has been introduced to account for the double counting of the phase space when the particles are identical. It is straightforward to show that this relation still holds when the two particles have different masses. From Eqs. (31), (32) the relationship between the states of infinite and asymptotically large (yet finite) volume can be deduced,

$$|i;\mathbf{P},E\rangle_\infty \Leftrightarrow 2\pi\sqrt{\frac{2V\rho_{V,i}E_0^*}{n_i q_i^*}}\,|i;\mathbf{P},E\rangle_V . \tag{33}$$

This relation therefore recovers the LL-factor as given in Eq. (30). It also demonstrates that the LL-factor accounts for different normalizations of the states in the finite volume and infinite volume in presence of interaction.

B. Two-Body LL-Factor in EFT($\not{\pi}$)

In this section we discuss the generalization of the LL-factor for the two-nucleon sector. This sector has been previously studied by Detmold and Savage [40] in the finite volume. They considered a novel idea of studying

electroweak matrix elements using a background field. They point out that evaluating matrix elements of electroweak currents between NN states, e.g. $< d|A^\mu|np >$, is naively one or two orders of magnitude more difficult than performing NN-four point functions. This argument led them to present a procedure for extracting the relevant LECs of the pionless EFT, EFT($\not\pi$) [43], by calculating four point functions of nucleons in a finite volume in the presence of a background electroweak field. This would be a project worth pursuing with great benefits, namely a five-point function is replaced by a four-point function, thereby dramatically reducing the number of propagator contractions. For isovector quantities, this procedure comes at a small cost, since for perturbatively small background fields the QCD generated gauge links get modified by a multiplicative factor that couples the valence quarks to the external field, $U_\mu^{QCD}(x) \to U_\mu^{QCD}(x)U_\mu^{ext}(x)$. On the other hand, for isoscalar quantities this approach would require the generation of gauge configurations in the presence of the background field. For both isovector and isoscalar quantities, one would need to perform calculations at a range of background field strengths in order to precisely discern the contribution of the coupling between the background field and the baryonic currents to the NN spectrum. Additionally, the nature of this background field will defer depending on the physics one is interested in calculating. Alternatively, one can always evaluate matrix elements of electroweak currents with gauge configuration that solely depend on the QCD action, which is the case considered here. In the following section, we present the relationship between the physical transition amplitude for the isovector-axial current and its FV matrix element. As mentioned above, there are great benefits for performing this calculation using the background field approach, however it is necessary to have further crosschecks.

1. ${}^1S_0 - {}^3S_1$ *mixing*

In this section, we will explore FV corrections of weak matrix elements in the two-nucleon sector. In particular, we will consider processes that mix the ${}^1S_0 - {}^3S_1$, e.g. proton-proton fusion $pp \to d + e^+ + \nu_e$. In order to do this calculation the mechanism of EFT($\not\pi$) [41], [42], [43] will be used. The methodology is similar to the one used for the toy problem considered in the first section, except the fields are now non-relativistic, and carry isospin and spin indices. The presence of a weak interactions, leads to a contribution to the Lagrangian that couples the axial-vector current $A^{\mu=3} = \frac{1}{2}\left(\bar{u}\gamma^3\gamma^5 u - \bar{d}\gamma^3\gamma^5 d\right)$ to an external weak current. In terms of our low-energy degrees of freedom, the axial current will receive one-body and two-body contributions. At energies well bellow the pion-production threshold, the EFT($\not\pi$) Lagrangian density including weak interactions can be written as [41], [42], [43]

$$\mathcal{L} = N^\dagger\left(i\partial_0 + \frac{\nabla^2}{2m} - \frac{W_3 g_A}{2}\sigma^3\tau^3\right)N - C_0^s\left(N^T P_1^j N\right)^\dagger\left(N^T P_1^j N\right) \tag{34}$$

$$-C_0^t\left(N^T P_3^j N\right)^\dagger\left(N^T P_3^j N\right) - W_3 L_{1,A}\left[\left(N^T P_1^3 N\right)^\dagger\left(N^T P_3^3 N\right) + h.c.\right] + \cdots \tag{35}$$

where N is the nucleon annihilation operator with bare mass m, $\{C_0^s, C_0^t, g_A, L_{1,A}\}$ are the LECs of the theory, $g_A = 1.26$ is the nucleon axial charge, W_3 is the external weak current, and $\{P_1^a, P_3^j\}$ are the standard $\{{}^1S_0, {}^3S_1\}$-projection operators,

$$P_1^a = \frac{1}{\sqrt{8}}\tau_2\tau^a\sigma_2 \qquad P_3^j = \frac{1}{\sqrt{8}}\tau_2\sigma_2\sigma^j, \tag{36}$$

where $\tau(\sigma)$ are the Pauli matrices which act in isospin(spin) space. In Eq. (34) the ellipsis denotes an infinite tower of higher order operators. The $\mathcal{O}(p^{2n})$-operator for the $\{{}^1S_0, {}^3S_1\}$ state will have a corresponding LEC $\{C_{2n}^s, C_{2n}^t\}$, which are included in this calculation. In the absence of weak interactions, the scattering amplitude for both channels can be determined exactly in terms of their corresponding LECs by performing a geometric series over all the bubble diagrams [43]

$$\mathcal{M}^0 = \frac{\sum C_{2n}p^{2n}}{1 - I_0^\infty \sum C_{2n}p^{2n}} \tag{37}$$

where the loop integral

$$I_0^\infty = \left(\frac{\mu}{2}\right)^{4-D}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{E - \frac{\mathbf{P}^2}{4m} - \frac{\mathbf{k}^2}{m} + i\epsilon} \tag{38}$$

is linearly divergent. When defined with the power-divergence subtraction (PDS) scheme [41, 42, 52], it becomes

$$I_0^\infty = -\frac{m}{4\pi}\left(\mu + i\sqrt{mE - \mathbf{P}^2/4}\right) = \frac{m}{4\pi}\left(\mu + iq^*\right). \tag{39}$$

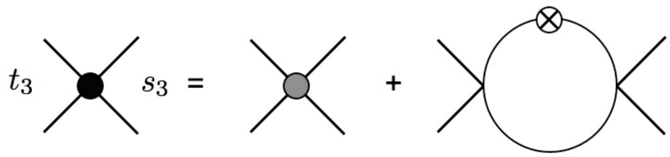

FIG. 2: Shown is the definition of the effective two-body coupling, $l_W^{\infty(V)}$ (represented by the black circle), between the singlet and triplet di-baryon states. The grey circle denotes the bare two-body weak coupling and the crossed circle is the one-body weak coupling.

where μ is the renormalization scale and as usual q^* is the momentum of each nucleon in the CM frame. It is well known that the NR scattering amplitude can be parametrized in terms of the phase shift, δ, and the CM momentum q^* as

$$\mathcal{M}^0 = \frac{4\pi}{mq^*}\frac{1}{\cot\delta - i} = \frac{4\pi}{mq^*}\frac{e^{i2\delta}-1}{2i} = \frac{4\pi}{mq^*}\frac{S-1}{2i}, \tag{40}$$

where we have used the fact that in the absence of weak interactions, the S-matrix is diagonal and equal to $e^{2i\delta}$. In the presence of the weak interaction, we use the "barred" parameterization in Eq. (1) to write the diagonal and off-diagonal scattering matrix elements

$$\mathcal{M}_{{}^1S_0({}^3S_1)} = \frac{4\pi}{mq^*}\frac{S_{11}-1}{2i} = \frac{4\pi}{mq^*}\frac{\cos 2\bar{\epsilon}\, e^{i2\delta_{{}^1S_0({}^3S_1)}}-1}{2i} \tag{41}$$

$$\mathcal{M}_{{}^1S_0-{}^3S_1} = \frac{4\pi}{mq^*}\frac{S_{12}}{2i} = \frac{4\pi}{mq^*}\frac{\sin 2\bar{\epsilon}\, e^{i\left(\delta_{{}^1S_0}+\delta_{{}^3S_1}\right)}}{2} \equiv |\mathcal{M}_{{}^1S_0-{}^3S_1}|\, e^{i\left(\delta_{{}^1S_0}+\delta_{{}^3S_1}\right)}, \tag{42}$$

In the absence of one-body contributions to the weak matrix elements, the determination of the QC for this system would be quite similar to our toy model in section I A. The mixing between the channels will lead to three coupled equations requiring the LECs to reproduce the infinite volume scattering matrix, thereby renormalizing the theory. In order to deal with the one-body operator, it is convenient to introduce a volume-dependent effective two-body coupling, $l_W^{\infty(V)}$, defined in Fig. (2). The regularization condition for this system at LO in the weak interaction can be written as

$$\sum_n C_{2n}^{{}^1S_0*} p^{2n} = \sum_n C_{2n}^{{}^1S_0} p^{2n} + \frac{(l^\infty)^2 I_0^\infty}{1-\sum_n C_{2n}^{{}^3S_1} p^{2n} I_0^\infty} = \frac{\mathcal{M}^0_{{}^1S_0}}{1+\mathcal{M}^0_{{}^1S_0} I_0^\infty}, \qquad W_3 l_W^\infty = \frac{\mathcal{M}_{{}^1S_0-{}^3S_1}}{(1+\mathcal{M}_{{}^1S_0} I_0^\infty)(1+\mathcal{M}_{{}^3S_1} I_0^\infty)}, \tag{43}$$

where we have introduced a volume dependent two-body coupling

$$l_W^{\infty(V)} \equiv L_{1,A} - g_A \frac{\mathcal{M}_{{}^1S_0}\mathcal{M}_{{}^3S_1} J_0^{\infty(V)}}{(1+\mathcal{M}_{{}^1S_0} I_0^\infty)(1+\mathcal{M}_{{}^3S_1} I_0^\infty)} \tag{44}$$

and the integral $J_0^{\infty,(V)}$ is defined as $J_0^{\infty,(V)}(E,\mathbf{P}) = -\partial_E I_0^{\infty,(V)}(E,\mathbf{P})$. The axial charge is fixed at its physical value. Once the LECs are renormalized, one can move on to evaluate the scattering amplitude for each channel in the FV. The poles of the scattering amplitude satisfy

$$\mathcal{R}e\left\{\left(\mathcal{M}_{{}^1S_0-{}^3S_1} - \mathcal{M}_{{}^1S_0}\mathcal{M}_{{}^3S_1} g_A W \delta J_0^V\right)^2 - \left(\mathcal{M}_{{}^1S_0} + \frac{1}{\delta I_0^V}\right)\left(\mathcal{M}_{{}^3S_1} + \frac{1}{\delta I_0^V}\right)\right\} = 0. \tag{45}$$

where $\delta J_0^V = J_0^V - J_0^\infty$ and $\delta I_0^V = I_0^V - I_0^\infty$ as before. In order to find the relation between δI_0^V and the Lüscher Zeta function, $\mathcal{Z}_{00}^0$, it is convenient to add and subtract the infinite volume two-particle propagator at zero energy in I_0^V. Then evaluate one of them using PDS and the other using the a hard cut-off, Λ, leading to

$$I^V(E,\mathbf{P}) = -\frac{m}{4\pi}\mu + \frac{1}{L^3}\sum_{\mathbf{k}}^{\Lambda} \frac{1}{E - \frac{\mathbf{P}^2}{4m} - \frac{\mathbf{k}^2}{m}} + \frac{m\Lambda}{2\pi^2} \tag{46}$$

therefore

$$\delta I^V(E,\mathbf{P}) = -\frac{mq^*}{4\pi}\left(\frac{\sqrt{4\pi}}{q^*\pi L}\mathcal{Z}_{00}^d\left[1;(q^*L/2\pi)^2\right] - i\right) \equiv -\frac{mq^*}{4\pi}\left(\cot(-\phi) - i\right) \tag{47}$$

At leading order in the mixing angle $\bar{\epsilon}$ and the axial coupling g_A Eq. (45) is in agreement with the result by Detmold *et al*. [40].

With this, it is straightforward to obtain the relationship between the FV matrix elements of the Hamiltonian density and the physical matrix element. In the absence of weak interactions, the two dibaryon states are degenerate with energy E_0 and momentum CM momentum q_0^*, satisfying the free quantization condition $\cot(-\phi) = \cot(\delta)$. As the weak interaction is turned on, the degeneracy is lifted, leading to a shift in energy equal to $\delta E = V|\mathcal{M}^V|$, where $|\mathcal{M}^V|$ is the FV matrix element of the Hamiltonian density between the ${}^1S_0$ and ${}^3S_1$ states. Expanding the Eq. (45) about the free energy, and keeping LO terms in the weak interaction, one obtains

$$\left(|\mathcal{M}^{\infty}_{{}^1S_0-{}^3S_1}| - g_A W_3 \frac{\delta J_0^V e^{i2\phi}}{\left(\delta I_0^V\right)^2}\right)^2 = \left(\frac{2\pi V}{q_0^{*2}}\right)^2 \left(\phi' + \delta'_{{}^3S_1}\right)\left(\phi' + \delta'_{{}^1S_0}\right)|\mathcal{M}^V_{{}^1S_0-{}^3S_1}|^2. \tag{48}$$

This result shows that in order to determine weak matrix elements in the nucleon sector, it is not only necessary to determine the derivatives of the phases shifts in the ${}^1S_0$ and ${}^3S_1$ channels, but it is also necessary to determine the axial charge. There is no clear crosscheck for this result, since the density of states approach cannot reproduce this. The presence of the one-body operator makes the mixing between the two states non-trivial, therefore one would expect a more complicated relationship between the FV and infinite volume states than the one predicted via the density of states approach. It must be mentioned that Eq. (48) does not depend on the value of W_3 since it factors out from both the FV and physical matrix elements.

2. $(BB)_I \to (BB)_{II}$

Lastly, we show how Eqs. (45), (48) can be extended to the low energy baryonic S-wave channels that couple exclusively via a weak tow-body interaction, which is pertinent for studying $\Lambda N \to NN$ and $H \to \Lambda N$ (with unphysical light-quark masses in order to assure the energy of the final states is bellow the pion production threshold). For these energies, one can generalize the EFT($\not{\pi}$) Lagrangian in Eq. (34) by replacing the nucleon field operator onto an SU(3) octet baryon field operator, B_i, with flavor index i. We can loosen the restrictions on the mass, and let it depend on the baryon flavor. Besides the spin and isospin structure, this is quite similar to our toy model in section I A. For generality, the channels will be labeled by the subscripts $\{I, II\}$. It turns out that this general case can be obtained from Eq. (45) in the limit $g_A \to 0$, after performing the following replacements:

$$m \to 2\mu_i = \frac{2}{\frac{1}{m_{i,1}} + \frac{1}{m_{i,2}}} \qquad q^* \to q_i^* = \sqrt{2\mu_i E - \frac{\mu_i^2 \mathbf{P}^2}{m_{i,1}m_{i,2}}}, \tag{49}$$

where $m_{i,j}$ is the mass of the j^{th} particle in the i^{th} channel with CM momentum q_i^*. With this, it is straight forward to show that the spectrum of this system satisfies

$$\mathcal{R}e\left\{|\mathcal{M}_{I-II}|^2 e^{i2(\delta_I+\delta_{II})} - \left(\mathcal{M}_I + \frac{1}{\delta I^V_{0,I}}\right)\left(\mathcal{M}_{II} + \frac{1}{\delta I^V_{0,II}}\right)\right\} = 0. \tag{50}$$

Using Eq. (47), one can show that the term inside the braces above is in fact real, and can be rewritten exactly as Eq. (24),

$$\cos 2\bar{\epsilon} \cos\left(\phi_I + \delta_I - \phi_{II} - \delta_{II}\right) = \cos\left(\phi_{II} + \delta_{II} + \phi_{II} + \delta_{II}\right). \tag{51}$$

The only difference between this result and Eq. (24) is the fact that for NR theories $\gamma = 1$ and the dispersion relationship differs its the relativistic counterpart.

Using the degenerate perturbation theory approach, the relationship between the FV matrix elements of the Hamiltonian density and the physical matrix element is found to be

$$\left|\mathcal{M}^{\infty}_{I-II}\right|^2 = \left(\frac{2\pi V}{q_I^{*2}}\right)\left(\frac{2\pi V}{q_{II}^{*2}}\right)\left(\phi'_{II} + \delta'_{II}\right)\left(\phi'_I + \delta'_I\right)|\mathcal{M}^V_{I-II}|^2. \tag{52}$$

In this case, this NR result for the LL-factor can be confirmed using the density of states approach. The FV four-point correlation function, Eq. (31), is the same for relativistic and NR theories. On the other hand, in the infinite volume

the phase space and the dispersion relation differ between the relativistic and non-relativistic theories. That being said, the NR four-point correlation function in infinite volume is

$$C^{\infty}_{NR,i}\left(t\right)=\frac{1}{4\pi^2}\int dE\;\frac{\mu_i}{2}q_i^*e^{-Et}\left|\langle 0|\,\sigma\left(\mathbf{0},0\right)|i;\mathbf{P},E\rangle_{NR,\infty}\right|^2. \tag{53}$$

Comparing this expression of the infinite volume correlation function to that of the asymptotically large FV results, Eq. (31), we find the relationship between the FV and infinite volume NR states

$$|i;\mathbf{P},E\rangle_{NR,\infty}\Leftrightarrow\sqrt{\frac{8\pi^2V\rho_{V,i}}{\mu_i q_i^*}}\,|i;\mathbf{P},E\rangle_{NR,V}\;,\quad\rho_{V,i}=\left(\phi_i'+\delta_i'\right)\frac{\mu_i}{4\pi q_i^*}, \tag{54}$$

which is consistent with Eq. (52).

III. SUMMARY AND CONCLUSION

In this paper, we have presented and derived the FV quantization condition for a system of multi-coupled channels each composed of two-hadrons in a moving frame. In the first section, the quantization condition at LO in mixing phase, Eq.(7) is derived for the s-wave scattering, where a rather simple EFT toy model for scalar particles is used. Using the techniques developed by Kim *et al.* [22], the quantization condition for boosted systems with arbitrary mixing angles, Eq.(15) has been obtained. It then became evident that the toy model used is in perfect agreement with the non-perturbative result when the series over angular momenta is truncated at the s-wave. From the generalized results, we have also derived the quantization condition for N=3 coupled channels composed of two hadrons, Eq.(26). This result has the advantage that it would allow the lattice practitioners to perform coupled channel calculations at multiple boosted momenta in a periodic lattice, thereby increasing the number of measurements in order to best constrain the S-matrix elements. When N=2, the S-matrix can be parametrized by three real parameters (two scattering phase shifts and one mixing angle), therefore one needs to perform at least three measurements at each CM momentum, which can be done by using combinations of different boost momenta and different volumes. The N=3 case would require six measurements for each CM momentum to constrain the three phase shifts and three mixing angles.

We have also derived the relationship between FV matrix elements and infinite volume matrix elements in the two-body sector, Eq.(30). This was first calculated for $K\to\pi\pi$ in the rest frame by Lellouch and Lüscher [39] and later extended to the boosted system by Kim *et al.* [22]. Here we showed two ways to obtain the extension of LL-factor for $2\to 2$ relativistic processes. The first entails expanding the coupled quantization, Eq.(7), about the free energy of the system when the two channels are decoupled. This method assumes the two states to be degenerate in the absence of the weak interaction, as was first developed by Lellouch and Lüscher [39]. Kim *et al.* [22] generalized the method of [51] and derived the relationship between FV and infinite volume two particle states in the moving frame, Eq.(33), which has been shown to agree with our result of the two-body relativistic LL-factor Eq.(30).

We have used EFT($\not\pi$) [41–43, 53, 54] to determine the extension of the LL-factor for non-relativistic baryonic systems. First processes that mix the ${}^1S_0-{}^3S_1$ NN channels have been considered. This is pertinent for performing calculations of proton-proton fusion, among other interesting processes, directly from LQCD, [40]. The channels in this system are mixed not only by a two-body operator but also by a one-body operator. As it is shown, FV effects arising from the insertion of a one-body operator are sizable and therefore must be included. It is convenient to introduce a volume-dependent effective mixing vertex that encompasses contribution from the one-body and two-body operators, Eq.(44). The effective vertex is required to reproduce the infinite volume transition amplitude. Expanding our quantization condition, Eq.(45), about the free energy solution, we find the relationship between the FV and infinite volume weak matrix elements, Eq.(48). Unlike any previous case, the FV and infinite volume weak matrix elements are not simply proportional to each other, and their relationship depends on the nucleon axial charge. We do not present a crosscheck for Eq.(48), because it is not clear how to extend the density of states approach in the presence of one-body operators. Eq.(48) allows one to determine physical weak matrix elements in the two-nucleon sector from LQCD, up to exponentially suppressed finite volume effects.

Finally, we have derived the NR coupled channel quantization condition for systems that do not have a one-body current operator contributing to the physics of interest, Eq.(50), which would be relevant for studying the H-dibaryon decays. This result is universal, and is independent of the spin/flavor composition of the two particle states. As in the previous case, the NR coupled-channels quantization condition can be expanded about the free energy solution to derive the LL-factor for $2\to 2$ NR processes in the absence of one-body contributions to the weak matrix elements,

Eq.(52). This result allows the practitioner to determine the dominant FV contributions arising when evaluating matrix elements of electroweak currents. In order to confirm this result, we derived the relation between the FV and infinite volume two particle state for NR systems using the density of states approach, Eq.(54). We found agreement between the two procedures, further confirming Eq.(52).

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